KINETICS OF TEMPERATURE FIELD IN A THREE-LAYERED COLLOIDAL BODY

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The net method is used to obtain the numerical solution of the problem of the kinetics of the temperature field in a three-layered colloidal body in the presence of phase transformations. The obtained results are compared with experimental data.

In some practically important problems of heat and mass transfer (purely technological problems of hydrothermal processing of materials, heat-engineering calculations of various kinds for barrier structures, automatic regulation of drying from the temperature of the material, etc.) only the kinetics of the temperature field needs to be investigated. The effect of mass transfer on the heat transfer in this case is corrected by the introduction into the differential heat conduction equations and boundary conditions of heat sources due to the occurring mass transfer processes and by the introduction of equivalent thermophysical coefficients.

With sufficient accuracy for practical purposes we can confine ourselves to approximate solutions obtained by numerical integration methods.

We consider a system composed of three unbounded contacting plates. We introduce the symbols

$$R_n = R_{n-1} + r_n \quad (n = 1,\ 2,\ 3;\quad R_0 = 0).$$

The thermophysical coefficients (TPC) of the plates in the general case depend on the temperature. The system of heat conduction equations is linearized by the introduction of averaged equivalent TPCs based on an analysis of the temperature dependence of the TPCs and an analysis of the kinetics of internal heat and mass transfer in the hydrothermal processing of material. The TPCs, like λ_n, for instance, are averaged by using the relationship

$$\bar{\lambda} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} \lambda(t)\, dt. \qquad (1)$$

The mathematical formulation of the problem is as follows:

$$\bar{c}_n \bar{\gamma}_n \frac{\partial T_n(x,\ \tau)}{\partial \tau} = \bar{\lambda}_n \frac{\partial^2 T_n(x,\ \tau)}{\partial x^2} + f_n(x,\ \tau),$$

$$(r_{n-1} \leqslant x \leqslant r_n \quad (n = 1,\ 2,\ 3);\quad 0 \leqslant \tau \leqslant \tau_K) \qquad (2)$$

with initial conditions

$$T_n(x,\ 0) = T_0 = \text{const}. \qquad (3)$$

At the contacting surfaces of the plates the boundary conditions are of the 4th kind

$$\bar{\lambda}_n \frac{\partial T_n(R_n - 0,\ \tau)}{\partial x} - \bar{\lambda}_{n+1} \frac{\partial T_{n+1}(R_{n+1} + 0,\ \tau)}{\partial x} = \psi_{n,n+1}(\tau)$$

$$(n = 1,\ 2), \qquad (4)$$

which express the occurrence of phase transformations and

$$T_n(R_n - 0,\ \tau) = T_{n+1}(R_n + 0,\ \tau) \quad (n = 1,\ 2). \quad (5)$$

On the outer boundaries of the plates the boundary conditions are of the 1st kind:

$$T_1(0,\ \tau) = \varphi_1(\tau);\quad T_3(R_3,\ \tau) = \varphi_3(\tau). \qquad (6)$$

An exact solution of (2) with more general boundary conditions, including boundary conditions of the 2nd and 3rd kind, is given in [1, 2].

Problem (2)–(6) is solved by the numerical method for the specific case of baking of wheat bread in a KhVK-2 oven (Odessa Bakery). The bread is regarded as a system of three unbounded plates in thermal contact [1, 3]. Subscripts n = 1, 3 indicate the upper and lower crusts, and n = 2 indicates the crumb. The thicknesses of the crusts and crumb are assumed to be constant. In fact, the thickness of the crusts during baking varies slightly owing to thermal diffusion of moisture and the gradual shift of the zone of evaporation to the center of the crumb. If we assume that the crusts are thin (in our case $r_1 = 0.002$ m, $r_3 = 0.003$ m) in comparison with the crumb ($r_2 = 0.070$ m) this assumption is quite valid for practical purposes. Moreover, the TPCs of the crusts were calculated as for a composite body (crust, dough) with due regard to the laws of crust formation [4].

The baking process, according to established theory, is divided into two periods: the periods of increasing and the constant rate of moisture removal [4].

Functions $f_n(x, \tau)$ in (2) take into account all the heat sources (the heat spent on the slight evaporation of moisture from the open surface and the effect of thermal diffusion of moisture on heat transfer in the first period, and the heating of molarly transferred vapor in the upper crust in the second period). Their analytical representation is based on a thorough analysis of the kinetics of moisture removal during baking, which is impossible in factory experiments.

In calculations of the temperature field of the dough—bread during baking the effect of mass transfer on the heat transfer was taken into account by the introduction of equivalent TCPs (the criterion K_λ, based on Ginzburg's investigations [4], for dough in the first period was taken as 0.14).

In the second period there is hardly any mass transfer in the crumb since the moisture content of the crumb is practically constant. In view of the above the values of $f_n(x, \tau)$ in the numerical integration of (2) are assumed to be zero. In (4) $\psi_{23}(\tau) = 0$ for the whole period of baking since almost all the moisture evaporated in the region of the lower crust moves into the crumb, where it condenses (the heats of evaporation and condensation are almost the same).

The evaporation of moisture on the upper crust/crumb boundary is significant only in the second period and was taken into account for this period only in (4) by means of a constant negative heat source $\psi_{12} = \mathrm{const} = C$.

For numerical integration of the simplified system (2)–(6) we used the rectangular net method. In the choice of the net equation the most important points are the stability in regard to rounding-off errors, the order of the approximation error, and its simplicity. In view of this we used a six-point symmetrical non-explicit net equation with an order of approximation $O(h^2 + l^2)$ [5, 6], which for an unbounded plate is written as follows:

$$-\frac{\bar{a}}{2}(t_{i-1,k+1} + t_{i+1,k+1}) + (\omega + \bar{a})\,t_{i,k+1} =$$

$$= \frac{\bar{a}}{2}(t_{i-1,k} + t_{i+1,k}) + (\omega - \bar{a})\,t_{i,k} \quad \begin{pmatrix} * & * & * \\ * & * & * \end{pmatrix}. \qquad (7)$$

This net equation is absolutely stable [7], which removes the limitations imposed on the relationship between h and l. For its solution we used the method of recursion [8], which is easily carried out by electronic computers.

The points of division of the media ($x = R_1$, $x = R_2$) are characterized by the fact that the temperature in their vicinity varies most. Hence, at the boundaries of contact of the media we have to use a net equation with a higher order of approximation but which still allows the use of the method of recursion. This can be done by the introduction of virtual nodes $(R_1 - h_2, kl)$, $(R_1 + h_1, kl)$ and $(R_2 - h_3, kl)$, $(R_2 + h_2, kl)$ $(k = 0, 1, 2, \ldots)$ at which the values of $t(x, \tau)$ are denoted by $t^*_{i_1-1,k}$, $t^*_{i_1+1,k}$ and $t^*_{i_2-1,k}$, $t^*_{i_2+1,k}$, respectively [6].

Assuming that the solution of problem (2)–(6) for one medium can be extrapolated smoothly into the neighboring medium (broken lines in Fig. 1) and writing the boundary condition (4) with an error not exceeding $O(h^4)$, and writing at the nodes $(R_1 - 0, (k+1)l)$ and $(R_1 + 0, (k+1)l)$ the more accurate equation with error $O(h^4)$ [6], we obtain formula (8) for calculation of the values of $t(x, \tau)$ at the point $x = R_1$, i.e., for $i = i_1$:

$$\frac{d_1(\omega_1 - 6\bar{a}_1)}{h_1} t_{i_1-1,k+1} + \frac{d_2(\omega_2 - 6\bar{a}_2)}{h_2} t_{i_1+1,k+1} +$$

$$+ \left[\frac{d_1(6\bar{a}_1 + 5\omega_1)}{h_1} + \frac{d_2(6\bar{a}_2 + 5\omega_2)}{h_2}\right] t_{i_1,k+1} =$$

$$= \frac{d_1(3\bar{a}_1 + \omega_1)}{h_1} t_{i_1-1,k} +$$

$$+ \frac{d_2(3\bar{a}_2 + \omega_2)}{h_2} t_{i_1+1,k} + \frac{3d_2\bar{a}_2}{h_2} t^*_{i_1-1,k} +$$

$$+ \frac{3d_1\bar{a}_1}{h_1} t^*_{i_1+1,k} - \left[\frac{d_1(6\bar{a}_1 - 5\omega_1)}{h_1} +\right.$$

$$\left. + \frac{d_2(6\bar{a}_2 - 5\omega_2)}{h_2}\right] t_{i_1,k} + 6C, \qquad (8)$$

where

$$d_1 = \bar{\lambda}_1/\bar{a}_1, \quad d_2 = \bar{\lambda}_2/\bar{a}_2.$$

The virtual values of $t^*_{i_1-1,k}$ and $t^*_{i_1+1,k}$ in (8) are calculated from the above net equations of higher accuracy.

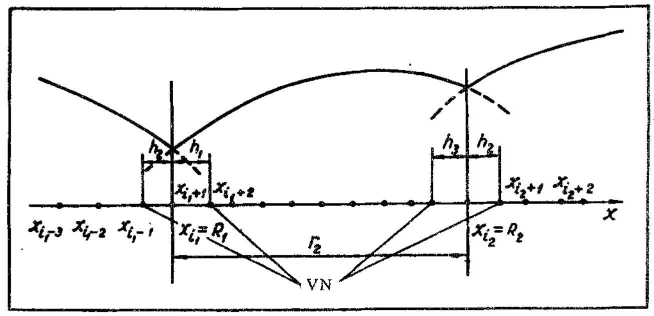

Fig. 1. Diagram of problem (VN are the virtual nodes).

For $\tau = 0$ (k = 0) the virtual values are determined from (3):

$$t^*_{i_1-1,0} = t_0 \text{ and } t^*_{i_1+1,0} = t_0.$$

There are similar formulas for the boundary of contact $x = R_2$, the only difference being that we must put $C = 0$ in the corresponding formulas since in (4) $\psi_{23}(\tau) = 0$.

To calculate the values of $t_{i,k+1}$ ($k = 1, 2, \ldots, n-1$) at points which do not belong to the boundaries of contact of the plates we use net equations of type (7) with error $O(h^2)$.

Thus, if on the k-th layer with respect to τ we know the values of $t_{i,k}$ ($i = 1, 2, \ldots, n-1$) and also the virtual values $t^*_{i_1-1,k}$, $t^*_{i_1+1,k}$ and $t^*_{i_2-1,k}$, $t^*_{i_2+1,k}$, then the (k + 1)th layer for points $x \neq R_1$, $x \neq R_2$ can be calculated from formulas of type (7) and for points $x = R_1$ and $x = R_2$ from formula (8) and similarly for the second boundary of contact. To calculate the (k + 1)th layer we obtain a system of linear equations with a Jacobian matrix, for the solution of which we use the method of recursion [6, 8].

The calculations were done on a Minsk-1 electronic digital computer for $h_1 = h_3 = 0.0001$ m, $h_2 = 0.001$ m; $l_1 = l_2 = l_3 = 3$ min.

The values used for the averaged equivalent TPCs for the first period of baking ($\tau = 18$ min) on the basis of our own and published data were: $\bar{\gamma}_1 = 450$ kg/m^3, $\bar{\lambda}_1 = 0.22$ W/m·deg, $\bar{a}_1 = 2.52 \cdot 10^{-7}$ m^2/sec, $\bar{\gamma}_2 = 550$ kg/m^3, $\bar{\lambda}_2 = 0.58$ W/m·deg, $\bar{a}_2 = 3.92 \cdot 10^{-7}$ m^2/sec, $\bar{\gamma}_3 = 500$ kg/m^3, $\bar{\lambda}_3 = 0.25$ W/m·deg, $\bar{a}_3 = 2.70 \cdot 10^{-7}$ m^2/sec. For the second period $\bar{\gamma}_1' = \bar{\gamma}_3' = 400$ kg/m^3, $\bar{\gamma}_2' = 450$ kg/m^3, $\bar{\lambda}_1 = \bar{\lambda}_1'$, $\bar{\lambda}_3 = \bar{\lambda}_3'$, $\bar{\lambda}_2' = 0.50$ W/m·deg, $\bar{a}_1' = 2.66 \cdot 10^{-7}$ m^2/sec, $\bar{a}_2' = 4.48 \cdot 10^{-7}$ m^2/sec. The specific heats calculated from these data for the crust, dough, and crumb agree with the most accurate published data [4].

The calculated and experimental data for the kinetics of the temperature field of bread during baking are given in Fig. 2.

The temperatures in the dough/bread were measured by copper-constantan differential thermocouples and recorded on graph tape in front of a calibrated electronic potentiometer. The thermocouple leads were wound helically between the fingers of cradles, which prevented their breakage.

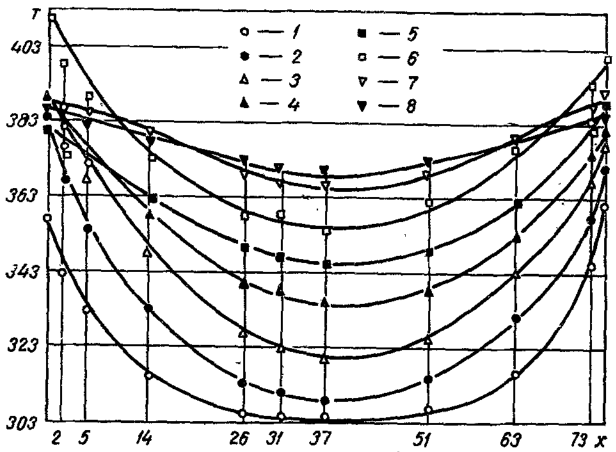

Fig. 2. Temperature fields of body during baking, obtained on Minsk-1 computer (continuous lines), and experimental data recorded on an electronic potentiometer (experimental points). Points 1–8 correspond to times $\tau = 6$, 12, 18, 24, 30, 36, 42, and 45 min from the start of baking. The height x of the bread is in mm, T is in ° K.

A comparison of the experimental and theoretical curves of the kinetics of the temperature field shows their good agreement. The somewhat higher value of the calculated temperature in the region of the upper crust must be attributed to the fact that in the solution, $f_1(x, \tau)$ and $\psi_{12}(\tau)$ were put equal to zero and, obviously, to other restrictions of the problem. In laboratory experiments $f_1(x, \tau)$ and $\psi_{12}(\tau)$ can be calculated on the basis of the kinetics of moisture removal.

NOTATION

r_n ($n = 1, 2, 3$) is the thickness of the plates, R_n are the total thicknesses of neighboring plates; $T(x, \tau)$ is the temperature; $\bar{c}_n$, $\bar{\gamma}_n$, $\bar{\lambda}_n$, and $\bar{a}_n$ are the equivalent averaged specific heat, density, thermal conductivity, and thermal diffusivity of the plates; τ_k is the duration of the hydrothermal treatment of the material; h and l are steps along space and time coordinates axes; $t(x, \tau)$ is the solution of the net equation; $\omega = h^2/l$ is a constant.

REFERENCES

1. A. A. Mikhelev, IFZh, no. 11, 1958.
2. V. P. Dushchenko and P. P. Lutsik, IFZh, no. 12, 1961.
3. A. A. Mikhelev, V. V. Zasyad'ko, A. T. Lisovenko, N. A. Tantsyura, and Yu. N. Zdorenko, IFZh, no. 10, 1960.
4. A. S. Ginzburg, Thermophysical Bases of the Baking Process [in Russian], Pishchepromizdat, 1955.
5. J. Crank and P. Nicolson, Proc. Cambr. Phil. Soc., 43, 50–67, 1947.
6. V. K. Saul'ev, Integration of Equations of the Parabolic Type by the Method of Nets [in Russian], FM, 1960.
7. G. O'Brien, M. Hyman and S. Kaplan, J. Math. and Phys., 29, no. 4, 223–251, 1951.
8. S. K. Godunov and V. S. Ryaben'kii, Introduction to the Theory of Difference Schemes [in Russian], FM, 1961.

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